

COMBINATORIAL HARMONIC COORDINATES

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ABSTRACT. Consider a planar, bounded, m -connected domain Ω and let $\partial\Omega$ be its boundary. Let \mathcal{T} denote a cellular decomposition of $\Omega \cup \partial\Omega$ where each 2-cell is either a triangle or a quadrilateral. We construct a *new* decomposition of $\Omega \cup \partial\Omega$ into \mathcal{R} , a finite, disjoint union of quadrilaterals. The construction is based on utilizing a *pair* of functions on $\mathcal{T}^{(0)}$ and properties of their level curves. The first function is obtained as the solution of a Dirichlet boundary value problem defined on $\mathcal{T}^{(0)}$. The second function is obtained by an integration scheme along the level curves of the first, and is called the *conjugate* function. It turns out that the pair of functions can be made *orthogonal* and *harmonic* in a combinatorial sense. As an application, we advance beyond the main results in [18].

0. INTRODUCTION

0.1. Perspective. In his attempts to prove uniformization Riemann suggested considering a planar annulus as made of a uniform conducting metal plate. When one applies voltage to the plate, keeping one boundary component at voltage k and the other at voltage 0, *electrical current* will flow through the annulus. The *equipotential* lines form a family of disjoint simple closed curves foliating the annulus and separating the boundary curves. The *current* flow lines consist of simple disjoint arcs connecting the boundary components, and they foliate the annulus as well. Together the two families provide “rectangular” coordinates on the annulus that turn it into a right circular cylinder or a (conformally equivalent) circular concentric annulus.

In [9], Brooks, Smith, Stone and Tutte studied square tilings of rectangles. They defined a correspondence between square tilings of rectangles and planar multigraphs endowed with two poles, a source and a sink. They viewed the multigraph as a network of resistors in which current is flowing. In their construction, a vertex corresponds to a connected component of the union of the horizontal edges of the squares in the tiling; an edge appears between two such vertices for each square whose horizontal edges lie in the corresponding connected components. Their approach is based on *Kirchhoff's circuit laws* which are widely used in the field of electrical engineering. It is interesting to recall that it was Dehn [14], who was the first to show (in 1903) a relation between square tiling and electrical networks. In [22], combining a mixture of geometry and probability, Kenyon used the fact that a resistor network corresponds to a reversible Markov chain to show a correspondence between a class of planar non-reversible Markov chains and trapezoid tilings. In [18] and [19], we addressed (using methods that transcend Dehn's ideas) the case where the domain has higher

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connectivity. This provides a first step towards an approximation of conformal maps from such domains onto a certain class of surfaces.

In [27], starting with a triangulation of a rectangle, Schramm employed discrete *extremal length* arguments and constructed a tiling of a rectangle by squares that preserves the contact graph of the given triangulation. In [11], Cannon, Floyd and Parry used extremal length arguments (similar to those in [27]) and gave a proof of a similar theorem. Another proof of a slight generalization of [27] was given by Benjamini and Schramm [7] (see also [8] for a related study). Both [27] and [11] are widely known as the “Finite Riemann Mapping Theorem,” which serves as the first step in Cannon’s “Combinatorial Riemann Mapping Theorem” (see [10]). We will comment about the relation of the above mentioned works to Theorem 0.4 in 3.1.1.

One may view the above mentioned theorems by Schramm, and by Cannon, Floyd, and Parry as attempts to provide various types of discrete versions of the conformal map obtained by the Riemann Mapping Theorem.

0.2. Combinatorial rectangular coordinates. In this paper, we will follow Riemann’s perspective on uniformization by constructing “rectangular” coordinates from given combinatorial data. The foundational modern theory of boundary value problems on graphs enables us to provide a unified framework to the theorems mentioned above, as well as to more general situations. The important work of Bendito, Carmona and Encinas (see for instance [4],[5] and [6]) is essential for our applications, and parts of it were utilized quite frequently in [17], [18], [19], this paper, and its sequel [20].

Consider a planar, bounded, m -connected domain Ω and let $\partial\Omega$ be its boundary. Henceforth, let \mathcal{T} denote a cellular decomposition of $\Omega \cup \partial\Omega$ where each 2-cell is either a triangle or a quadrilateral. We will construct a *new* decomposition of $\Omega \cup \partial\Omega$ into \mathcal{R} , a finite, disjoint union of quadrilaterals. We will show that the space of quadrilaterals can be endowed with a finite measure thought of as a combinatorial analogue of the Euclidean planar area measure.

Next, we construct a pair (S_Ω, f) where S_Ω is a special type of a genus 0 *singular flat surface* having m boundary components, which is tiled by rectangles and is endowed with μ , the canonical area measure induced by the singular flat structure. The map f is a homeomorphism from $(\Omega, \partial\Omega)$ onto S_Ω . Furthermore, each quadrilateral is mapped to a single rectangle and its measure is preserved.

The proof that f is a homeomorphism as well as the construction of a measure on the space of quadrilaterals depend in a crucial way on constructing a *pair* of harmonic functions on $\mathcal{R}^{(0)}$ and a few properties of their level curves.

Let g denote the solution of a Dirichlet boundary value problem defined on $\mathcal{T}^{(0)}$ (see Definition 1.4). We will start by extending g to the interior of the domain: affinely over edges in $\mathcal{T}^{(1)}$ and over triangles $\mathcal{T}^{(2)}$, and bilinearly over quadrilaterals in $\mathcal{T}^{(2)}$. We will often abuse notation and will not distinguish between a function defined on $\mathcal{T}^{(0)}$ and its extension over $|\mathcal{T}|$.

For the applications of this paper, particularly in creating “rectangular” coordinates in a combinatorial and topological sense, it is necessary to define a new function θ on $\mathcal{T}^{(0)}$. This function will be defined on an annulus minus a slit (or a quadrilateral) and will be called the *conjugate function*. It is obtained by integrating the *normal derivative* of g along its level curves (Definition 1.1). The normal derivative of g is initially defined only at vertices

that belong to $\partial\Omega$. The simple topological structure of the level curves of g permits the extension of the normal derivative to the interior, and thereafter its integration. In the case of an annulus, these level curves are simple, piecewise linear closed curves that separate the boundary curves and foliate the annulus. Definition 2.6 will formalize this discussion.

The analysis of the level curves of θ is the subject of Proposition 2.21. Their interaction with the level curves of g is described in Proposition 2.22. In the case of an annulus, the level curves of g form a piecewise-linear analogue of the level curves of the function $u(r, \phi) = r$, and those of θ form a piecewise-linear analogue of the level curves of the function $v(r, \phi) = \phi$.

We now turn to a more technical description of this paper.

A *simple* quadrilateral is a closed topological disk with precisely four vertices on its boundary. For $h : \mathcal{T}^{(0)} \rightarrow \mathbb{R}$ recall that $dh(e) = h(e^+) - h(e^-)$ for any oriented edge $e = [e^-, e^+]$. A *slit* in an annulus is a fixed, simple, combinatorial path in $\mathcal{T}^{(1)}$ along which g is monotone increasing which joins the two boundary components (Definition 2.1).

We now make

Definition 0.1 (A pair of combinatorial orthogonal functions). Let $(\Omega, \partial\Omega, \mathcal{T})$ be given, where Ω is an annulus minus a slit. A pair of non-negative functions ϕ and ψ defined on $\mathcal{T}^{(0)}$ will be called *combinatorially orthogonal*, if there exist a cellular decomposition \mathcal{R} of $\Omega \cup \partial\Omega$, where each 2-cell is a simple quadrilateral, and extensions of ϕ and ψ to $\mathcal{R}^{(0)}$, such that

$$(0.2) \quad d\phi(e)d\psi(e) = 0, \text{ for every } e \in \mathcal{R}^{(1)}.$$

Such a decomposition will be called a *rectangular combinatorial net*.

We record the properties of the pair which are essential for the applications of this paper.

Theorem 0.3. *The pair $\{g, \theta\}$ is combinatorially orthogonal. Furthermore, there is a choice of conductance constants on $\mathcal{R}^{(1)}$ such that both functions are combinatorially harmonic.*

0.3. Geometric models and maps. We now turn to describing some applications of the pair $\{g, \theta\}$ and the rectangular net \mathcal{R} . In the course of the proofs of our main theorems, we will first construct a new decomposition of Ω into a rectangular net, \mathcal{R} , then a model surface which is, when $m > 2$, a singular flat surface tiled by rectangles. Finally we will construct a map between the domain and the model surface and describe its properties..

Let us start with the fundamental case, an annulus. Given two positive real numbers r_1 and r_2 and two angles $\phi_1, \phi_2 \in [0, 2\pi)$, a bounded domain in the complex plane whose boundary is determined by the two circles $u(r, \phi) = r_1$ and $u(r, \phi) = r_2$ and the two radial curves $v(r, \phi) = \phi_1$ and $v(r, \phi) = \phi_2$, will be called an *annular shell*. Let μ denote Lebesgue measure in the plane. In the statement of the next theorem, the measure ν which is described in Definition 3.1, is determined by g and θ . The quantity $\text{period}(\theta)$ is an invariant of θ which encapsulates integration of the normal derivative of g along its level curves (see Definition 2.19).

Our first main theorem is

Theorem 0.4 (A Dirichlet problem on an annulus). *Let \mathcal{A} be a planar annulus endowed with a cellular decomposition \mathcal{T} , and let $\partial\mathcal{A} = E_1 \sqcup E_2$. Let k be a positive constant and let g be the solution of the Dirichlet boundary value problem defined on $(\mathcal{A}, \partial\mathcal{A}, \mathcal{T})$.*

Let S_A be the concentric Euclidean annulus with its inner and outer radii satisfying

$$(0.5) \quad \{r_1, r_2\} = \{1, 2\pi \text{Length}(L(v_k))\} = \{1, 2\pi \exp\left(\frac{2\pi}{\text{period}(\theta)} k\right)\}.$$

Then there exist

- (1) a tiling T of S_A by annular shells,
- (2) a homeomorphism

$$f : (\mathcal{A}, \partial\mathcal{A}, \mathcal{R}) \rightarrow (S_A, \partial S_A, T),$$

such that f maps each quadrilateral in $\mathcal{R}^{(2)}$ onto a single annular shell in S_A , f preserves the measure of each quadrilateral, i.e.,

$$\nu(R) = \mu(f(R)), \text{ for all } R \in \mathcal{R}^{(2)},$$

and f is boundary preserving.

The dimensions of each annular shell in the tiling are determined by the boundary value problem (in a way that will be described later). In our setting, boundary preserving means that the annular shell associated to a quadrilateral in \mathcal{R} with an edge on $\partial\Omega$ will have an edge on a corresponding boundary component of S_A .

Our second main theorem is Theorem 4.4. It provides a geometric model for the case $m > 2$. The model surface that generalizes the concentric annulus in the previous theorem first appeared in [18]. It is a singular flat, genus zero compact surface with $m > 2$ boundary components with conical singularities. It is called a *ladder of singular pairs of pants*.

We obtain a topological decomposition of Ω into simpler objects, annuli and generalized annuli, for which the previous theorem and a slight generalization of it may be applied. The second step of the proof is geometric. We show that it is possible to glue the pieces along common boundaries in a *length* preserving way. This entails a new notion of length which will be introduced in Definition 2.36.

0.4. Advances beyond [18]. From [18] we use the description of the topological properties of singular level curves of the Dirichlet boundary value problem. The most significant one is a description of the topological structure of the connected components of any singular level curve of the solution. A study of the topology and geometry of the associated level curves and their complements is carried out in [18, Section 2].

The advances of this paper beyond [18] are due to the fact that we are able to make a *change of charts* on Ω . That is, we view the given cellular decomposition \mathcal{T} as a set of initial charts. Any component of the complement of a singular level curve is either an annulus or an annulus with one singular boundary component. We obtain a rectangular net on it using level curves of $\{g, \theta\}$.

Once this step is accomplished, we are able to advance beyond the main results of [18] as well as the other works mentioned in Subsection 0.1. We will show that the map we construct from the domain to a model surface is a *homeomorphism* onto its image, and furthermore that it preserves a combinatorial two-dimensional measure.

Recall that in the theorems proved in [18] (as well as in [19]), the analogous mapping to f was proved to be an *energy-preserving* map (in a discrete sense) from $\mathcal{T}^{(1)}$ onto a particular singular flat surface. It is not possible to extend that map to a homeomorphism defined on

the domain. Furthermore, the natural invariant measure considered there is one-dimensional (being concentrated on edges).

In [20], we will continue our work by showing that a scheme of refining the cellular decomposition, coupled with a particular choice of a conductance function in each step, leads to convergence of the mappings constructed in each step, to a canonical *conformal* mapping from the domain onto a model surface.

0.5. Organization of the paper. A modest familiarity with [18] will be useful for reading this paper. For the purpose of making this paper self-contained, a few basic definitions and some notations are recalled in Section 1, and results from [18] are quoted as needed. In Section 2, the main technical tools of this paper, a conjugate function and a rectangular net, are constructed on an annulus minus a slit. In Section 3, the cases of an annulus and an annulus with one singular boundary component are treated, respectively, by Theorem 0.4 and Proposition 3.22. Due to the reasons we mentioned above, these are foundational for the applications of this paper and of [20] as well. Section 4 is devoted to the proof of Theorem 4.4.

Convention. In this paper we will assume that a fixed affine structure is imposed on $(\Omega, \partial\Omega, \mathcal{T})$. The existence of such a structure is obtained by using *normal coordinates* on $(\Omega, \partial\Omega, \mathcal{T})$ (see [29, Theorem 5-7]). Since our methods depend on the combinatorics of the triangulation, the actual chosen affine structure is not important.

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1. FINITE NETWORKS AND BOUNDARY VALUE PROBLEMS

In this section we briefly review classical notions from harmonic analysis on graphs through the framework of *finite networks*. We then describe a procedure to modify a given boundary problem and \mathcal{T} . The reader who is familiar with [18] or [19] may skip to the next section.

1.1. Finite networks. In this paragraph we will mostly be using the notation of Section 2 in [3]. Let $\Gamma = (V, E, c)$ be a planar *finite network*; that is, a planar, simple, and finite connected graph with vertex set V and edge set E , where each edge $(x, y) \in E$ is assigned a *conductance* $c(x, y) = c(y, x) > 0$. Let $\mathcal{P}(V)$ denote the set of non-negative functions on V . Given $F \subset V$ we denote by F^c its complement in V . Set $\mathcal{P}(F) = \{u \in \mathcal{P}(V) : S(u) \subset F\}$, where $S(u) = \{x \in V : u(x) \neq 0\}$. The set $\delta F = \{x \in F^c : (x, y) \in E \text{ for some } y \in F\}$ is called the *vertex boundary* of F . Let $\bar{F} = F \cup \delta F$ and let $\bar{E} = \{(x, y) \in E : x \in F\}$. Let $\bar{\Gamma}(F) = (\bar{F}, \bar{E}, \bar{c})$ be the network such that \bar{c} is the restriction of c to \bar{E} . We write $x \sim y$ if $(x, y) \in \bar{E}$.

The following operators are discrete analogues of classical notions in continuous potential theory (see for instance [16] and [13]).

Definition 1.1. Let $u \in \mathcal{P}(\bar{F})$. Then for $x \in F$, the function

$$(1.2) \quad \Delta u(x) = \sum_{y \sim x} c(x, y) (u(x) - u(y))$$

is called the *Laplacian* of u at x . For $x \in \delta(F)$, let $\{y_1, y_2, \dots, y_m\} \in F$ be its neighbors enumerated clockwise. The *normal derivative* of u at a point $x \in \delta F$ with respect to a set F is

$$(1.3) \quad \frac{\partial u}{\partial n}(F)(x) = \sum_{y \sim x, y \in F} c(x, y)(u(x) - u(y)).$$

Finally, $u \in \mathcal{P}(\bar{F})$ is called *harmonic* in $F \subset V$ if $\Delta u(x) = 0$, for all $x \in F$.

1.2. Boundary value problems on graphs. Consider a planar, bounded, m -connected region Ω , and let $\partial\Omega$ be its boundary ($m > 1$). Let \mathcal{T} be a cellular decomposition of $\Omega \cup \partial\Omega$, where each cell is either a triangle or a quadrilateral. Let $\partial\Omega = E_1 \sqcup E_2$, where E_1 is the outermost component of $\partial\Omega$. Invoke a conductance function \mathcal{C} on $\mathcal{T}^{(1)}$, thus making it a finite network, and use it to define the Laplacian on $\mathcal{T}^{(0)}$.

We will now describe the type of boundary value problem defined on $\mathcal{T}^{(0)}$ that will be employed in this paper.

Definition 1.4. The *Dirichlet* boundary value problem (D-BVP) is determined by requiring that

- (1) $g|_{E_1} = k$, $g|_{E_2} = 0$, and
- (2) $\Delta g = 0$ at every interior vertex of $\mathcal{T}^{(0)}$.

These data will be called *Dirichlet data* for Ω .

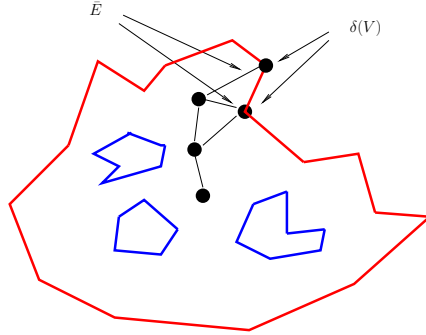


FIGURE 1.5. An example of a possible splitting scheme.

A fundamental property which we often will use is the *discrete maximum-minimum principle*, asserting that if u is harmonic on $V' \subset V$, where V is a connected subset of vertices having a connected interior, then u attains its maximum and minimum on the boundary of V' (see [28, Theorem I.35]).

The following proposition (cf. [3, Prop. 3.1]) establishes a discrete version of the first classical *Green identity*. It played an important role in the proofs of the main theorems in [17, 18], and it also plays an important role in this paper and in its sequel [20].

Proposition 1.6 (The first Green identity). *Let $F \subset V$ and $u, v \in \mathcal{P}(\bar{F})$. Then we have that*

$$(1.7) \quad \sum_{(x,y) \in \bar{E}} c(x, y)(u(x) - u(y))(v(x) - v(y)) = \int_{x \in F} \Delta u(x)v(x) + \int_{x \in \delta(F)} \frac{\partial u}{\partial n}(F)(x)v(x).$$

1.3. Piecewise-linear modifications of a boundary value problem. We will often need to modify a given cellular decomposition, and thereafter to modify the initial boundary value problem. The need to do this is twofold. First assume, for example, that L is a fixed, simple, closed level curve of the initial boundary value problem. Since $L \cap \mathcal{T}^{(1)}$ is not (generically) a subset of $\mathcal{T}^{(0)}$, Definition 2.36 may not be employed directly to provide a notion of length to L . Therefore, we will add vertices and edges according to the following procedure. Such new vertices will be called type I vertices.

Let $\mathcal{O}_1, \mathcal{O}_2$ be the two distinct connected components of the complement of L in Ω , with L being the boundary of both (these properties follow by employing the Jordan curve theorem). We will call \mathcal{O}_1 an *interior domain* if all the vertices which belong to it have g -values that are smaller than the g -value of L . The other domain will be called the *exterior domain*. Note that by the maximum principle, one of $\mathcal{O}_1, \mathcal{O}_2$ must have all of its vertices with g -values smaller than the g -value of L .

Let $e \in \mathcal{T}^{(1)}$ and assume that $x = e \cap L$ is a vertex of type I. Thus, two new edges (x, v) and (u, x) are created. We may assume that $v \in \mathcal{O}_1$ and $u \in \mathcal{O}_2$. Next, define conductance constants $\tilde{c}(v, x)$ and $\tilde{c}(x, u)$ by

$$(1.8) \quad \tilde{c}(v, x) = \frac{c(v, u)(g(v) - g(u))}{g(v) - g(x)} \quad \text{and} \quad \tilde{c}(u, x) = \frac{c(v, u)(g(u) - g(v))}{g(u) - g(x)}.$$

By adding to \mathcal{T} all such new vertices and edges, as well as the piecewise arcs of L determined by the new vertices, we obtain two cellular decompositions, $\mathcal{T}_{\mathcal{O}_1}$ of \mathcal{O}_1 and $\mathcal{T}_{\mathcal{O}_2}$ of \mathcal{O}_2 . Two conductance functions, $\mathcal{C}_{\mathcal{O}_1}$ and $\mathcal{C}_{\mathcal{O}_2}$ are now defined on the one-skeleton of these cellular decompositions, by modifying according to Equation (1.8) the conductance constants that were used in the Dirichlet data for g (i.e., changes are occurring only on new edges, and on L the conductance is defined to be identically zero). One then defines (see [18, Definition 2.7]) a natural modification of the given boundary value problem, the solution of which is easy to control by using the existence and uniqueness theorems in [3]. In particular, it is equal to the restriction of g to \mathcal{O}_i , for $i = 1, 2$.

Another technical point which motivates the modification described above will manifest in Subsection 2.3. Proposition 1.6 will be frequently used in this paper, and it may not be directly applied to a modified cellular decomposition and the modified boundary value problem defined on it. Formally, in order for Proposition 1.6 to be applied, the modified graph of the network needs to have its vertex boundary components separated enough in terms of the combinatorial distance. Whenever necessary, we will add new vertices along edges and change the conductance constants along new edges in such a way, that the solution of the modified boundary value problem will still be harmonic at each new vertex and will preserve the values of the solution of the initial boundary value problem at the two vertices along the original edge. Such vertices will be called type II vertices.

Formally, once such changes occur, a new Dirichlet boundary value problem is defined. The existence and uniqueness of the solution of a Dirichlet boundary value problem (see [3]) allow us to abuse notation and keep denoting the new solution by g .

2. CONSTRUCTING A COMBINATORIAL NET AND HARMONIC CONJUGATE FUNCTION ON AN ANNULUS WITH A SLIT

This section has three subsections. The first subsection contains the construction of the conjugate function to the initial Dirichlet boundary value problem defined on an annulus. The second subsection supplies the proof of Theorem 0.3. Finally, in the third subsection, we will define the pair-flux metric and its induced length. These notions will be essential to the proof of Theorem 4.4 in which gluing two components of the complement of a singular level curve of the solution occurs.

2.1. Constructing the conjugate function. In this subsection, we will construct a function which is conjugate (in a combinatorial sense) to g , the solution of a Dirichlet boundary value problem defined on an annulus. The conjugate function will be single valued on the annulus minus a chosen *slit*.

With the notation of the previous section and the introduction, let $(\mathcal{A}, \partial\Omega = E_1 \cup E_2, \mathcal{T})$ be an annulus endowed with a cellular decomposition in which each 2-cell is either a triangle or a quadrilateral. Let k be a positive constant, and let g be the solution of a Dirichlet boundary value problem as described in Definition 1.4. All the level curves of g are piecewise, simple closed curves separating E_1 and E_2 (see Lemma 2.8 in [18] for the analysis in the case of higher connectivity) which foliate \mathcal{A} .

Before providing the definition of the conjugate function to g , we need to make a choice of a piecewise linear path in \mathcal{A} .

Definition 2.1. Let $\text{slit}(\mathcal{A})$ denote a fixed, simple combinatorial path in $\mathcal{T}^{(1)}$ which joins E_1 to E_2 . Furthermore, we require that the restriction of the solution of the boundary value problem to it is monotone increasing.

Remark 2.2. The existence of such a path is guaranteed by the maximum principle.

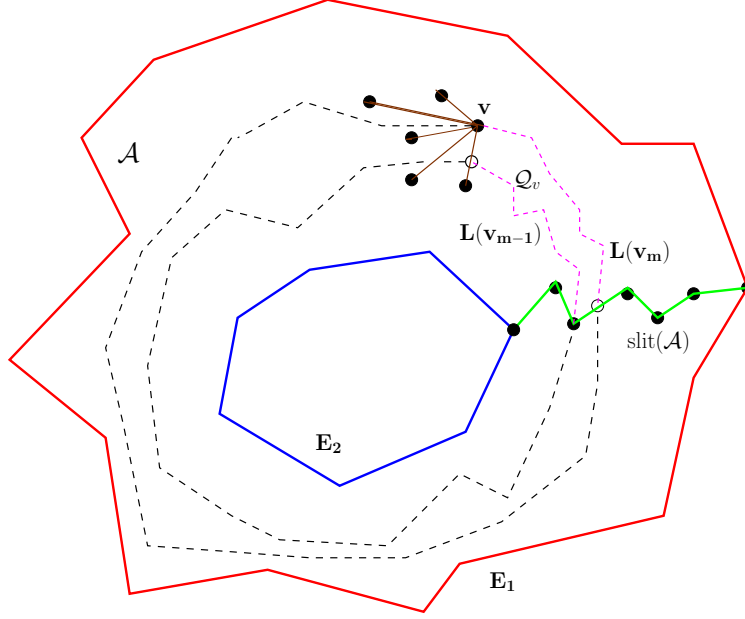
Let

$$(2.3) \quad \mathcal{L} = \{L(v_0), \dots, L(v_k)\}$$

be the collection of level curves of g that contain all the vertices in $\mathcal{T}^{(0)}$ arranged according to increasing values of g . It follows from Definition 1.4 that $L(v_0) = E_2$ and $L(v_k) = E_1$.

In order to obtain a single valued function, we will start with a preliminary case. To this end, let $\mathcal{Q}_{\text{slit}}$ denote the quadrilateral obtained by cutting open \mathcal{A} along $\text{slit}(\mathcal{A})$ and having two copies of $\text{slit}(\mathcal{A})$ attached. Since in \mathcal{A} orientation is well defined, we will denote one of the two copies by $\partial\mathcal{Q}_{\text{base}}$ and the other by $\partial\mathcal{Q}_{\text{top}}$. In other words, from the point of view of \mathcal{A} , points on $\text{slit}(\mathcal{A})$ may be endowed with two labels, recording whether they are the starting point of a level curve (with winding number equals to one) or its endpoint. We keep the values of g at the vertices unchanged. Thus, corresponding vertices in $\partial\mathcal{Q}_{\text{base}}$ and $\partial\mathcal{Q}_{\text{top}}$ have identical g -values. By abuse of notation, we will keep denoting by $\mathcal{T}^{(0)}$ the 0-skeleton of $\mathcal{Q}_{\text{slit}}$.

For $v \in \mathcal{A} \setminus E_2$, which is in $\mathcal{T}^{(0)}$ or a vertex of type I, let $L(v_m)$ denote the unique level curve of g which contains v . Let \mathcal{Q}_v be the piecewise-linear quadrilateral whose boundary is defined by $\text{slit}(\mathcal{A})$, $L(v_m)$, $L(v_{m-1})$ and the leftmost edge emanating from v with its endpoint on $L(v_{m-1})$.

FIGURE 2.4. An example of a quadrilateral Q_v .

For $v \in \mathcal{T}^{(0)} \cap E_2$ recall that $E_2 = L(v_0)$ is the unique level curve of g which contains v . Let \bar{Q}_v be the piecewise-linear quadrilateral whose boundary is defined by $\text{slit}(\mathcal{A})$, E_2 , $L(v_1)$ and the rightmost edge emanating from v with its endpoint on $L(v_1)$.

Remark 2.5. Note that while Q_v was defined in \mathcal{A} , it is a well defined object in Q_{slit} as well.

We are now ready to present

Definition 2.6 (The Conjugate function). Let v be a vertex in $\mathcal{T}^{(0)}$ or a vertex of type I. Let

$$(2.7) \quad \pi(v) = L(v) \cap \partial Q_{\text{base}}.$$

We define $\theta(v)$, the conjugate function of g , as follows.

First case: Suppose that $v \notin E_2$. Then

$$(2.8) \quad \theta(v) = \int_{\pi(v)}^v \frac{\partial g}{\partial n}(Q_v)(u),$$

where the integration is carried along $L(v)$ in the counter-clockwise direction.

Second case: Suppose that $v \in E_2$. Then we define $\theta(v)$ by

$$(2.9) \quad \theta(v) = \int_{\pi(v)}^v \left| \frac{\partial g}{\partial n}(\bar{Q}_v)(u) \right|,$$

where the integration is carried along E_2 in the counter-clockwise direction. For a point $z \in Q_{\text{slit}}$ which is not a vertex as above, $\theta(z)$ is defined by extending θ affinely over edges and triangles, and bi-linearly over quadrilaterals.

Remark 2.10. The reasons for using $\bar{Q}(v)$, the “rightmost” and the absolute value of the normal derivative of g in the second case, are due to the orientation along E_2 and the maximum principle, respectively. Furthermore, the continuity of θ from the right on E_2 follows from similar arguments to those appearing in the proof of Proposition 2.13.

Remark 2.11. It is worth mentioning that the modification of the definition to the case of a quadrilateral, \mathcal{Q} , is immediate. One observes that once a Dirichlet-Neumann boundary value problem is imposed (see [19]), the level curves of the solution are disjoint, piecewise-linear simple curves that foliate the quadrilateral and join its bottom to its top (see [19, Proposition 2.1]). Hence, the definition above would apply once we define $\text{slit}(\mathcal{Q})$ to be the bottom of \mathcal{Q} ($\partial\mathcal{Q}_{\text{bottom}}$), and then the analogue to \mathcal{Q}_v is defined as above.

Remark 2.12. Note that we abuse notation and keep denoting by E_1 and by E_2 the two boundary components of $\mathcal{Q}_{\text{slit}}$, which naturally correspond to their counterparts in \mathcal{A} .

We now turn to studying the topological structure of the level curves of θ .

By definition, $\partial\mathcal{Q}_{\text{base}}$ is the level curve of θ which corresponds to $\theta = 0$. We will prove that $\partial\mathcal{Q}_{\text{top}}$ is also a level curve of θ . In other words, computing the value of θ at the endpoint of a level curve emanating from $\partial\mathcal{Q}_{\text{base}}$ is independent of the level curve chosen. The proof is an application of the first Green identity (see Proposition 1.6).

Proposition 2.13. *The curve $\partial\mathcal{Q}_{\text{top}}$ is a level curve of θ in $\mathcal{Q}_{\text{slit}}$.*

Proof. Let L_1 and L_2 be any two level curves of g which start at $\partial\mathcal{Q}_{\text{base}}$ and have their endpoints x_1 and x_2 on $\partial\mathcal{Q}_{\text{top}}$, respectively. Without loss of generality, assume that the g -value of L_2 is bigger than the g -value of L_1 . We must show that

$$(2.14) \quad \theta(x_1) = \theta(x_2).$$

First, add vertices of type II according to the procedure defined in Subsection 1.3, so that the first Green identity, Proposition 1.6, may be applied to the network which is defined on the quadrilateral determined by $L_1, \partial\mathcal{Q}_{\text{top}}, L_2$ and $\partial\mathcal{Q}_{\text{base}}$. Let us denote this quadrilateral by $\mathcal{Q}_{(L_1, L_2)}$.

Let $h \equiv 1$ be the constant function defined in $\mathcal{Q}_{(L_1, L_2)}$. The assertion of Proposition 1.6, applied with the functions h and g on the induced network, yields

$$(2.15) \quad \int_{x \in \partial\mathcal{Q}_{(L_1, L_2)}} \frac{\partial g}{\partial n}(\mathcal{Q}_{(L_1, L_2)})(x) = 0.$$

However, since g is harmonic in \mathcal{A} and in particular along $\text{slit}(\mathcal{A})$, we have that

$$(2.16) \quad \int_{x \in \partial\mathcal{Q}_{\text{top}}} \frac{\partial g}{\partial n}(\mathcal{Q}_{(L_1, L_2)})(x) + \int_{x \in \partial\mathcal{Q}_{\text{base}}} \frac{\partial g}{\partial n}(\mathcal{Q}_{(L_1, L_2)})(x) = 0.$$

Therefore, we must have that

$$(2.17) \quad \int_{x \in L_1} \frac{\partial g}{\partial n}(\mathcal{Q}_{(L_1, L_2)})(x) + \int_{x \in L_2} \frac{\partial g}{\partial n}(\mathcal{Q}_{(L_1, L_2)})(x) = 0.$$

Since g is harmonic in \mathcal{A} , it follows easily that

$$(2.18) \quad \int_{x \in L_1} \frac{\partial g}{\partial n}(\mathcal{Q}_{E_2, L_1})(x) + \int_{x \in L_1} \frac{\partial g}{\partial n}(\mathcal{Q}_{(L_1, L_2)})(x) = 0.$$

Equation (2.14) now follows from (2.17), (2.18) and the definition of θ (Definition 2.6). \square

As a consequence of this proposition we can now make

Definition 2.19. The *period* of θ is defined by

$$(2.20) \quad \text{period}(\theta) = \int_{u \in L(v_m)} \frac{\partial g}{\partial n}(\mathcal{Q}_{E_2, L(v_m)})(u), \text{ where } m \in \{0, \dots, k\}.$$

We continue the study of the level curves of θ . Note that the maximum principle, and its definition, θ is monotone increasing along level curves of g . This property will now be used in the following

Proposition 2.21. *Each level curve of θ has no endpoint in the interior of $\mathcal{Q}_{\text{slit}}$, is simple, and joins E_1 to E_2 ($\partial\mathcal{Q}_{\text{bottom}}$ to $\partial\mathcal{Q}_{\text{top}}$ in the quadrilateral case). Furthermore, any two level curves of θ are disjoint.*

Proof. Suppose that a level curve of θ which starts at $s \in E_2$ has an endpoint ξ in $T \in \mathcal{T}^{(2)}$, where T lies in the interior of \mathcal{A} . Let $[s, \xi]$ be the intersection of this level curve with the interior of \mathcal{A} . Let L_ξ denote the level curve of g that passes through ξ . Since the level curves of g foliate \mathcal{A} , there exists a level curve L_ψ of g , which is as close as we wish to L_ξ , and such that its intersection with $[s, \xi]$ is empty. Since θ is monotone increasing and continuous along L_ψ , it assumes all values between 0 and $\text{period}(\theta)$. Hence, it will assume the value $\theta(\xi)$. This shows that no level curve of θ can have an interior endpoint.

Assume that one of the level curves of θ is not simple. Let \mathcal{D} be any domain which is bounded by it. Since the level curves of g foliate the annulus, one of these intersects the boundary of \mathcal{D} in at least two points. The monotonicity of θ along the level curves of g renders this impossible.

Assume that there exists a level curve of θ , $L(\theta)$ which does not join E_1 to E_2 . By construction, each level curve of θ does not have an endpoint inside \mathcal{A} and its intersection with each 2-cell is a segment (or a point). Hence, both endpoints of $L(\theta)$ must lie on E_1 or on E_2 . Without loss of generality, assume that both endpoints are on E_1 . Hence, there must be a level curve of g that intersects $L(\theta)$ in at least two points. This is impossible again.

The fact that level curves of θ that correspond to the same value may not intersect each other follows from similar arguments to those appearing the first part of the proof. \square

Of special importance is the interaction between the level curves of θ and the level curves of g . The following proposition will show that, from a topological point of view, the union of the two families of level curves resemble a planar coordinate system. This proposition is the topological prerequisite for the proof of Theorem 0.3, which will appear in the next subsection.

Proposition 2.22. *The intersection number between any level curve of θ and any level curve of g is equal to 1.*

Proof. It easily follows for the proof of Proposition 2.21 that the intersection number of any level curve of g with any level curve of θ is at most equal to 1. Since both families of level curves foliate $\mathcal{Q}_{\text{slit}}$, the intersection number is equal to 1. \square

2.2. Proof of Theorem 0.3. We now turn to the proof of Theorem 0.3. As required, we will construct \mathcal{R} and then prove the existence of conductance constants defined on $\mathcal{R}^{(1)}$ so that *both* g and θ are harmonic. It will follow from the construction that Equation (0.2) (see Definition 0.1) holds.

Proof. Each vertex in $\mathcal{T}^{(0)}$ or a vertex of type I in $\mathcal{Q}_{\text{slit}}$ belongs to one and only one of the level curves of θ . Let

$$(2.23) \quad \mathcal{V} = \{L(\theta_0), \dots, L(\theta_m)\}$$

be the set of level curves of θ which contain all of the vertices mentioned above. Note that by definition $L(\theta_0) = \partial\mathcal{Q}_{\text{base}}$ is the level set of θ which corresponds to the value 0, and that by Proposition 2.13, $L(\theta_m)$ is the level curve of θ which corresponds to the value $\text{period}(\theta)$.

Let \mathcal{R} be the cellular decomposition induced by the union of \mathcal{V} and \mathcal{L} . It follows from Proposition 2.22 that each 2-cell in \mathcal{R} is a quadrilateral. Note that $\mathcal{R}^{(0)}$ contains more vertices than the ones used to define the families \mathcal{L}, \mathcal{V} . To end the proof of the theorem we now prove

Proposition 2.24. *There exists a choice of conductance constants on $\mathcal{R}^{(1)}$ so that both g and θ are harmonic.*

Proof. The proof is a direct consequence of Proposition 2.22. It implies that each interior vertex in $\mathcal{R}^{(0)}$, has exactly two edges that are parts of a level curve of g and exactly two edges that are parts of a level curve of θ , emanating from it.

Let $v(i, j)$ denote the vertex which is the intersection of $L(v_i)$ and $L(\theta_j)$, with $i = 1, \dots, (k-2)$, $j = 0, \dots, (m-2)$. We define, starting with $i = 1$ and $j = 1$,

$$(2.25) \quad c(v_{i,j}, v_{i,j+1}) \text{ and } c(v_{i,j}, v_{i,j-1})$$

to be any two positive constants such that

$$(2.26) \quad c(v_{i,j}, v_{i,j+1})(\theta(v_{i,j}) - \theta(v_{i,j+1})) + c(v_{i,j}, v_{i,j-1})(\theta(v_{i,j}) - \theta(v_{i,j-1})) = 0.$$

Choosing such constants is possible since θ is monotone increasing in the counter clockwise direction along $L(v_i)$.

Similarly, we define

$$(2.27) \quad c(v_{i-1,j}, v_{i,j}) \text{ and } c(v_{i,j}, v_{i+1,j})$$

to be any two positive constants such that

$$(2.28) \quad c(v_{i-1,j}, v_{i,j})(g(v_{i,j}) - g(v_{i-1,j})) + c(v_{i+1,j}, v_{i,j})(g(v_{i,j}) - g(v_{i+1,j})) = 0.$$

Choosing such constants is possible because g is monotone increasing along every level set of θ .

Each level set of g or of θ is homeomorphic to a closed interval. Hence, the choices made above can successively be continued along the level curves of g and θ until the second to last edges, $[v_{i,m-1}, v_{i,m}], i = 1, \dots, (k-2)$ and $[v_{k-1,m-1}, v_{k-1,m}], m = 1, \dots, (m-2)$, respectively. For the very last edges, compatible conductance constants are determined, by the all the choices that were made before.

A simple computation now shows that with respect to these choices of conductance constants on $\mathcal{R}^{(1)}$, both g and θ are harmonic.

Proposition 2.24

Finally, since each edge $e \in \mathcal{R}^{(1)}$ is part of either a level set of g or a level set of θ , it is evident that Equation (0.2) holds. This ends the proof of the theorem. □

Remark 2.29. It follows from Proposition 2.13 that the net R can be extended to a rectangular net on \mathcal{A} .

It follows from the proof that there is a fair amount of flexibility in the choice conductance constants. For the application of this paper, a particular choice hereby will be made—one that keeps the L_1 norm of the normal derivative of g with respect to $\mathcal{T}^{(0)}$ unchanged.

Lemma 2.30. *There is a choice of conductance constants on $\mathcal{R}^{(1)}$ such that both g and θ are harmonic, and furthermore*

$$(2.31) \quad \int_{\mathcal{T}^{(0)} \cap E_1} \frac{\partial g}{\partial n}(v) = \int_{\mathcal{R}^{(0)} \cap E_1} \frac{\partial g}{\partial n}(u).$$

Proof. In light of the previous proposition, it suffices to show that one can choose conductance constants

$$(2.32) \quad c(v_{k-1,j}, v_{k,j}), \quad j = 1, \dots, m-1,$$

so that Equation (2.31) holds, and then proceed as before.

Suppose that

$$(2.33) \quad g|_{L(v_{k-1})} = r.$$

It then easily follows by setting (for example)

$$(2.34) \quad c(v_{k-1,j}, v_{k,j}) = \frac{\int_{\mathcal{T}^{(0)} \cap E_1} \frac{\partial g}{\partial n}(v)}{(m-1)(k-r)}, \quad j = 1, \dots, m-1$$

that Equation (2.31) holds. □

Remark 2.35. The assertion of this proposition means that computing $\text{period}(\theta)$ (see Definition 2.19), with respect to $\mathcal{T}^{(0)}$ endowed with the conductance constants given in the Dirichlet data for g or with respect to $\mathcal{R}^{(0)}$ endowed with the conductance constants chosen above, yields the same number.

2.2.1. *Viewing θ from a PDE perspective.* The term “conjugate” associated with θ is motivated by the following properties of θ with respect to \mathcal{R} and $\mathcal{Q}_{\text{slit}}$:

- θ is harmonic,
- $\theta|_{\mathcal{Q}_{\text{base}}} = 0$, $\theta|_{\mathcal{Q}_{\text{top}}} = \text{period}(\theta)$, and
- $\frac{\partial \theta}{\partial n}|_{E_1 \cup E_2} = 0$.

Hence, θ satisfies the combinatorial analogues of the analytical properties of the polar angle function $v(r, \phi) = \phi$ in the complex plane, which is known to be, when it is single-value defined, the harmonic conjugate function of $v(r, \phi) = r$.

2.2.2. *Related work.* Our definition of the harmonic conjugate function is motivated by the fact that, in the smooth category, a conformal map is determined by its real and imaginary parts, which are known to be harmonic conjugates. The search for discrete approximation of conformal maps has a long and rich history. We refer to [24] and [12, Section 2] as excellent recent accounts.

Convincing evidence that a discrete scheme may converge to the Riemann mapping is supplied, for example, by the solution to a conjecture suggested by Thurston and based on the *Circle Packing Theorem* by Koebe-Andreiev-Thurston. Thurston conjectured that a certain discrete scheme will converge to the Riemann mapping. The conjecture, which was proved in 1987 by Rodin and Sullivan (see [26]), provides a refreshing and geometric view on Riemann’s Mapping Theorem.

We should also mention that a search for a combinatorial Hodge star operator has recently gained much attention and is closely related to the construction of a harmonic conjugate function. We refer the reader to [21] and to [25] for further details and examples for such operators..

2.3. **The pair-flux length.** In this subsection, we will define a notion of *length* for the level curves of g and of θ . To give some perspective, recall that in [15], Duffin defined a metric to be a function $\tau : E \rightarrow [0, \infty]$. More recently, in [10], Cannon defined a *discrete metric* to be a function $\rho : V \rightarrow [0, \infty)$. The length of a path is then given by integrating τ, ρ along it, respectively.

In [18, Definition 1.9], we defined a metric (in Cannon’s sense) which utilized g , the solution of the boundary value problem, alone.

Motivated by the planar Riemannian case (see Equation (2.41)), we will define new notions of metric and length for level curves of g in $\mathcal{Q}_{\text{slit}}$ and \mathcal{Q} , respectively. These notions will incorporate both g and its conjugate function, θ .

Definition 2.36. With the notation of the previous sections we define the following.

- (1) For $e = [e^-, e^+]$, let $\psi(e) = e^-$ be the map which associates to an edge its initial vertex. The *pair-flux weight* of e is defined by
- (2.37)

$$\rho(e) = \frac{2\pi}{\text{period}(\theta)} \exp\left(\frac{2\pi}{\text{period}(\theta)} g(\psi(e))\right) |d\theta(e)| = \frac{2\pi}{\text{period}(\theta)} \exp\left(\frac{2\pi}{\text{period}(\theta)} g(e^-)\right) |d\theta(e)|.$$

- (2) Let L be any path in \mathcal{R} ; then its length with respect to the pair-flux weight is given by integrating ρ along it,

$$(2.38) \quad \text{Length}(L) = \int_{e \in L} \rho(e).$$

In the applications of this paper, we will use the pair-flux weight to provide a notion of length to level curves of g . Thus, by the assertion of Proposition 2.13, we may now deduce

Corollary 2.39. *Let $L(v)$ be a closed level curve of g , and let $0 \leq m = g|_{L(v)} \leq k$; then we have*

$$(2.40) \quad \text{Length}(L(v)) = 2\pi \exp\left(\frac{2\pi}{\text{period}(\theta)} m\right).$$

The definition of the pair-flux length is one of the new advances of this paper. Whereas in [18, 19, 20] other notions of lengths utilizing only the solution g were introduced, the pair-flux length incorporates the pair $\{g, \theta\}$. This appealing feature is also the case in the smooth category, i.e., for $z = r \exp(i\theta)$ in the complex plane, we have

$$(2.41) \quad dz = r \exp(i\theta) d\theta + \exp(i\theta) dr.$$

We will now provide a notion of length to the level curves of θ in $\mathcal{Q}_{\text{slit}}$. Keeping the analogy with the planar Riemannian case, the restriction of the Euclidean length element to level curves of the function $v(r, \phi) = \phi_0$, has the form

$$(2.42) \quad |dz| = dr.$$

Definition 2.43. Let $L(\theta) = (v_0, \dots, v_k)$ be a level curve of θ with $v_0 \in E_2$ and $v_k \in E_1$, then its length is given by

$$(2.44) \quad \text{Length}(L(\theta)) = \exp(g(v_k)) - \exp(g(v_0)) = \exp(k) - 1.$$

Remark 2.45. It is a consequence of part (2) of Proposition 2.22 that any two level curves of θ have the same length.

3. THE CASES OF AN ANNULUS AND AN ANNULUS WITH ONE SINGULAR BOUNDARY COMPONENT

3.1. The case of an annulus. The setting of this subsection is a special case of the one described in Definition 1.4. Let \mathcal{R} be the cellular decomposition into quadrilaterals associated with the orthogonal pair $\{g, \theta\}$ which was constructed in the proof of Theorem 0.3.

We use the term *measure* on the space of quadrilaterals in $\mathcal{R}^{(2)}$ to denote a non-negative set function defined on $\mathcal{R}^{(2)}$. An example of such, which will be used in Theorem 0.4, is provided in

Definition 3.1. For any $R \in \mathcal{R}^{(2)}$, let $R_{\text{top}}, R_{\text{base}}$ be the top and base boundaries of R , respectively. Let $t \in R_{\text{top}}^{(0)}$ and $b \in R_{\text{base}}^{(0)}$ be any two vertices. Then we let

$$(3.2) \quad \nu(R) = \frac{1}{2} \left(\exp^2\left(\frac{2\pi}{\text{period}(\theta)} g(t)\right) - \exp^2\left(\frac{2\pi}{\text{period}(\theta)} g(b)\right) \right) \frac{2\pi d\theta(R_{\text{base}})}{\text{period}(\theta)}.$$

Remark 3.3. By construction, all the vertices in R_{top} (R_{base}) have the same g values and $d\theta(R_{\text{base}}) = d\theta(R_{\text{top}})$.

We now turn to the

Proof of Theorem 0.4. Recall (see the discussion preceding Proposition 2.24) that the vertices in $\mathcal{R}^{(0)}$ are comprised of all the intersections of the level curves of g (the family \mathcal{L}) and the level curves of θ (the family \mathcal{V}). The vertex (i, j) will denote the unique vertex determined by the intersection of $L(v_i)$ and $L(\theta_j)$ (the existence and uniqueness of this intersection are consequences of Proposition 2.22).

The conjugate function θ is single-valued on $\mathcal{Q}_{\text{slit}}$ and multi-valued with a period which is equal to $\text{period}(\theta)$, when extended to \mathcal{A} . This means that

$$(3.4) \quad \theta(z_1) = \theta(z_0) + \text{period}(\theta),$$

whenever $z_1 \in L(z_0)$ is obtained from $z_0 \in \text{slit}(\mathcal{A})$ by traveling one full cycle along the level curve $L(z_0)$. Hence, the discrete function

$$(3.5) \quad \frac{2\pi}{\text{period}(\theta)} (g(v) + i\theta(v)), \quad v \in \mathcal{A} \cap \mathcal{R}^{(0)}$$

has period $2\pi i$ when defined on \mathcal{A} . Therefore,

$$(3.6) \quad \exp\left(\frac{2\pi}{\text{period}(\theta)} (g(v) + i\theta(v))\right), \quad v \in \mathcal{A} \cap \mathcal{R}^{(0)}$$

is single-valued on \mathcal{A} .

We now turn to the construction of the tiling T .

The tiling T of $S_{\mathcal{A}}$ is determined by all the intersections of the family of concentric circles, \mathcal{C} , defined by

$$(3.7) \quad r_i = \exp\left(\frac{2\pi}{\text{period}(\theta)} g(v_i)\right), \quad \text{for } i = 0, \dots, k,$$

with the family of radial lines Γ , defined by

$$(3.8) \quad \phi_j = \frac{2\pi}{\text{period}(\theta)} \theta_j, \quad \text{for } j = 0, \dots, m,$$

where each annular shell in the tiling is uniquely defined by four vertices that lie on two consecutive members of the families above.

Let $f_{\mathcal{R}}$ be the homeomorphism which sends the quadrilateral \mathcal{R} determined by the counterclockwise oriented ordered set of vertices

$$(3.9) \quad \{(i, j), (i+1, j), (i+1, j+1), (i, j+1)\}$$

for $i = 0, \dots, k$ and $j = 0, \dots, m$, onto the annular shell T_R determined by the counterclockwise oriented ordered set of vertices

$$(3.10) \quad \{r_i \exp(i\phi_j), r_{i+1} \exp(i\phi_j), r_{i+1} \exp(i\phi_{j+1}), r_i \exp(i\phi_{j+1})\}$$

and that preserves the order of the vertices.

By the definitions of ν, μ and T_R , we have for all $R \in \mathcal{R}^{(2)}$ that

$$(3.11) \quad \nu(R) = \mu(T_R).$$

Let

$$(3.12) \quad f = \bigcup_{R \in \mathcal{R}^{(2)}} f_R.$$

Then it is clear from (3.9) and (3.10) that f maps any edge in $\mathcal{R}^{(1)} \cap \partial\Omega$ homeomorphically onto an arc in $\partial S_{\mathcal{A}}$. Therefore, f is indeed boundary preserving.

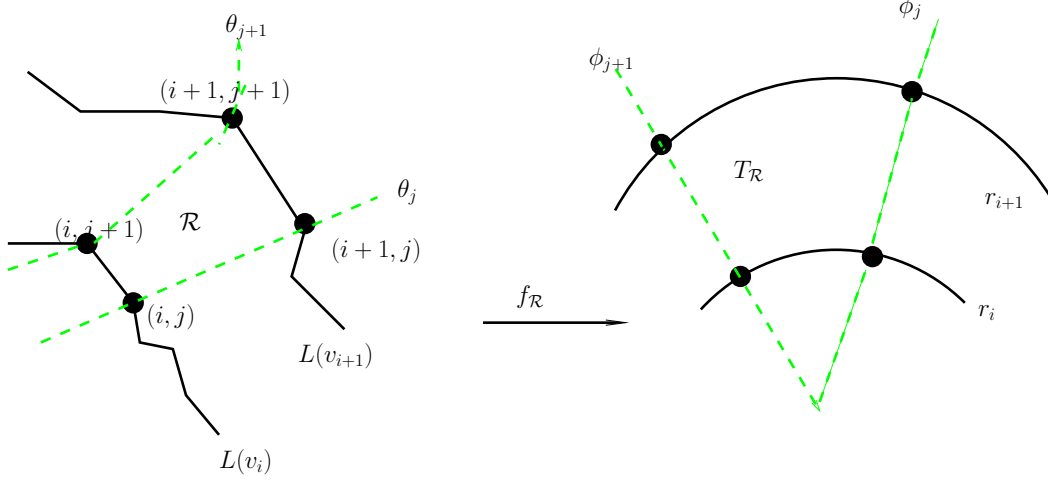


FIGURE 3.13. Constructing one annular shell.

It remains to prove that the map $f = \cup_R f_R$ assembled from the individual maps defined above, which is clearly a homeomorphism onto its image, is onto $S_{\mathcal{A}}$. To this end, observe that by the maximum principle, the map f is into $S_{\mathcal{A}}$.

It is clear from the construction of \mathcal{R} that any two quadrilaterals in $\mathcal{R}^{(2)}$ have disjoint interiors and that their intersection is either a single vertex or a common edge. Also recall that by definition each quadrilateral has its top and bottom edges situated on two successive level curves in \mathcal{L} , and its right and left edges situated on two successive level curves in \mathcal{V} . Since the union of the quadrilaterals in $\mathcal{R}^{(2)}$ tile \mathcal{A} , the total ν -measure of their union, which we define to be $\nu(\mathcal{A})$, satisfies

$$(3.14) \quad \nu(\mathcal{A}) \equiv \nu\left(\bigcup_{R \in \mathcal{R}^{(2)}} R\right) = \sum_{R \in \mathcal{R}^{(2)}} \nu(R).$$

Starting from the quadrilaterals that lie between $L(v_0)$ and $L(v_1)$, we sum the ν -measure of all the quadrilaterals in the layer defined in between successive level curves of g , until we reach $L(v_k)$. An easy computation employing Definition 3.1 now shows that

$$(3.15) \quad \nu(\mathcal{A}) = \pi \left(\exp^2 \left(\frac{2\pi}{\text{period}(\theta)} k \right) - 1 \right).$$

By the construction of the annular shells and the definition of the map f , each quadrilateral R is mapped onto a unique annular shell T_R , no two different quadrilaterals are mapped onto the same annular shell, and the collection of their images tiles a subset of $S_{\mathcal{A}}$.

Hence, by applying the above paragraph, (3.14), (3.11), (3.15) and the definition of $S_{\mathcal{A}}$, we obtain that

$$(3.16) \quad \nu(\mathcal{A}) = \sum_{R \in \mathcal{R}^{(2)}} \mu(T_R) = \mu\left(\bigcup_{R \in \mathcal{R}^{(2)}} T_R\right) = \mu(S_{\mathcal{A}}).$$

Hence, there are no gaps nor overlaps in the tiling of $S_{\mathcal{A}}$. This concludes the proof of the theorem. □

Remark 3.17. The proof further shows that in fact level curves of g are mapped homomorphically onto the level curves $u(r, \phi) = r_i$, and level curves of θ are mapped homeomorphically onto the level curves $v(r, \phi) = \phi_j$.

3.1.1. Relation to works by Schramm and Cannon-Floyd-Parry. It seems imperative to relate this theorem to Theorem 1.3 in [27] and Theorem 3.0.1 in [11]. While Schramm, and by Cannon, Floyd, and Parry used discrete extremal lengths arguments in their proofs, the arguments as well as the results are a bit different. Schramm's proof seems to work for a quadrilateral but not directly for an annulus. The methods of Cannon, Floyd and Parry work for both a quadrilateral and an annulus. Furthermore, Schramm's input is a triangulation with a contact graph that will (more or less) be preserved. The input for Cannon, Floyd and Parry is more flexible. They consider a combinatorial of a topological quadrilateral (annulus) by topological disks. We refer the reader to the papers above for details. Upon applying a Dirichlet-Neumann boundary value problem, our proof will work for the quadrilateral as well.

It is showed in [27, page 117] that if one attempts to use the combinatorics of the hexagonal lattice, square tilings (as provided by Schramm's method) cannot be used as discrete approximations for the Riemann mapping. There is still much effort by Cannon, Floyd and Parry to provide sufficient conditions under which their method will converge to a conformal map.

While our proof of Theorem 0.4 does not use the machinery of extremal length arguments, it is worth recalling that in the smooth category there are celebrated connections between boundary value problems and extremal length (see for example [1, Theorem 4.5]).

The common theme of our methods and those of Cannon, Floyd and Parry in [11] is the construction of a new coordinate system on a topological annulus. As stated in the introduction, this idea goes back to Riemann.

3.2. The case of an annulus with one singular boundary component. In this subsection we will generalize Theorem 0.4 by providing a geometric model for an annulus with one singular boundary component. The singular boundary component is of a special type. It is determined by the topological structure of a singular level curve of the solution of a Dirichlet boundary value problem imposed on a planar embedded m -connected domain, where $m > 1$.

We start with two definitions; the first one appeared in [18, page 9].

Definition 3.18. A generalized bouquet circles will denote a union of bouquets of piecewise-linear circles where the intersection of any two circles is at most a vertex. Moreover, all such tangencies are required to be exterior, i.e., no circle is contained in the interior of the bounded component of another.

Recall that Theorem 2.15, which was proved in [18], asserts the following.

Theorem 3.19 (The topology of a level curve). *Let L be a connected level curve for g . Then each connected component of L is a generalized bouquet of circles.*

It is convenient to present the singular boundary component as a quotient space. In the following definition, a circle will mean either a round circle or a piecewise linear circle.

To this end we make

Definition 3.20. An embedded planar circle with finitely many distinguished points on it, will be called a *labeled* circle. If in addition, equivalence relations among these points is given, so that the quotient of the labeled circle is a generalized bouquet of circles, then we call the quotient a *labeled bouquet*, and the labeled circle will be called *good*.

Remark 3.21. Note that if a labeled round bouquet, i.e., one which consists of only round circles, has more than two round circles tangent at one point, it will no longer embed in \mathbb{R}^2 .

We will now define the object of study in this subsection. By a *generalized singular annulus*, $\mathcal{A}_{\text{sing}}$, we will mean a subset of the plane, whose interior is homeomorphic to the interior of an annulus, and whose boundary has two components: one of which is homeomorphic to S^1 and the other is a generalized bouquet of circles. The subscript denotes the set of tangency points in the generalized bouquet of circles. Let us also assume that a cellular decomposition \mathcal{T} of $(\mathcal{A}_{\text{sing}}, \partial\mathcal{A}_{\text{sing}})$ is provided, where each 2-cell is either a triangle or a quadrilateral.

Topologically, $\mathcal{A}_{\text{sing}}$ may be presented as the quotient of a planar annulus \mathcal{A} , where $\partial\mathcal{A} = E_1 \cup E_2$, and E_2 is a good labeled circle (see Definition 3.20). Henceforth, we will let π denote the quotient map. We will let \dot{E}_2 denote the singular boundary component of $\partial\mathcal{A}_{\text{sing}}$.

Note that the cellular decomposition \mathcal{T} can be lifted to a cellular decomposition $\tilde{\mathcal{T}}$ of $(\mathcal{A}, \partial\mathcal{A})$, where each 1-cell, 2-cell in $\tilde{\mathcal{T}}$, respectively, is the unique pre-image, under π^{-1} , of a unique 1-cell, 2-cell in \mathcal{T} , respectively. The difference between the two cellular decompositions manifests in the addition (in comparison to \dot{E}_2) of vertices in E_2 . Specifically, for each vertex v in the singular part of $\partial\mathcal{A}_{\text{sing}}$, there are $m(v)$ vertices in E_2 , where $m(v)$ is the number of circles that are tangent at v .

We will now apply Theorem 0.3 and Theorem 0.4 to $(\mathcal{A}, \partial\mathcal{A}, \tilde{\mathcal{T}})$. In the following proposition, recall that the existence of \mathcal{R} is provided by Theorem 0.3, and that θ is the conjugate harmonic function to g , the solution of the imposed Dirichlet boundary value problem on $(\mathcal{A}, \partial\mathcal{A}, \tilde{\mathcal{T}})$.

With the above notation and setting in place, and with \mathcal{L} denoting the set of level curves of g as in equation (2.3), we may now state the main proposition of this subsection.

Proposition 3.22 (An annulus with a singular boundary). *Let $(\mathcal{A}_{\text{sing}}, \partial\mathcal{A}_{\text{sing}} = E_1 \cup \dot{E}_2)$ be a generalized singular annulus endowed with a cellular decomposition \mathcal{T} . Let k be a positive constant and let g be the solution of the Dirichlet boundary value problem defined on $(\mathcal{A}, \partial\mathcal{A}, \tilde{\mathcal{T}})$.*

Let $S_{\mathcal{A}}$ be the concentric Euclidean annulus with its inner and outer radii satisfying

$$(3.23) \quad \{r_1, r_2\} = \{1, 2\pi \text{Length}(L(v_k))\} = \{1, 2\pi \exp\left(\frac{2\pi}{\text{period}(\theta)} k\right)\}.$$

Then there exist

- (1) *a tiling T of $S_{\mathcal{A}}$ by annular shells,*
- (2) *a set denoted by $\text{sing}(S_{\mathcal{A}})$ consisting of finitely many points which is contained in the inner boundary of $S_{\mathcal{A}}$,*
- (3) *a homeomorphism*

$$f : (\mathcal{A}, \partial\mathcal{A} \setminus \pi^{-1}(\text{sing}(\mathcal{A})), \mathcal{R}) \rightarrow (S_{\mathcal{A}}, \partial S_{\mathcal{A}} \setminus \text{sing}(S_{\mathcal{A}}), T)$$

such that f maps the interior of each quadrilateral in $\mathcal{R}^{(2)}$ onto the interior of a single annular shell in $S_{\mathcal{A}}$, f preserves the measure of each quadrilateral, i.e.,

$$\nu(R) = \mu(f(R)), \text{ for all } R \in \mathcal{R}^{(2)},$$

and f is boundary preserving.

Proof. The proof is a straightforward modification of the non-singular boundary case. Let \mathcal{R} be the rectangular net constructed in Theorem 0.3. Let θ be the conjugate harmonic function constructed on $(\mathcal{A}, \partial\mathcal{A}, \tilde{\mathcal{T}})$, and let f be the map constructed in Theorem 0.4. Let T be the tiling of $S_{\mathcal{A}}$ provided by Theorem 0.4.

For each $t_i \in \text{sing}(\mathcal{A}) \subset \dot{E}_2$, $i = 1, \dots, p$, there are precisely $m(t_i)$ vertices on E_2 in the equivalence class corresponding to t_i . Let

$$(3.24) \quad \mathcal{V}(t_i) = \{L(\theta)_{t_{i,1}}, \dots, L(\theta)_{t_{i,m(t_i)}}\}, \quad i = 1, \dots, p$$

be the level curves of θ that have one of their endpoints at these vertices. With this notation, and since the level curves of θ are parallel, it follows that

$$(3.25) \quad \mathcal{V}_{\text{sing}(\mathcal{A})} = \bigcup_{i=1}^p \mathcal{V}(t_i)$$

comprises of all the level curves of θ that have an endpoint in the pre-image of $\text{sing}(\mathcal{A})$.

Set

$$(3.26) \quad \text{sing}(S_{\mathcal{A}}) = f(E_2) \cap f\left(\bigcup_{i=1}^p \mathcal{V}(t_i)\right),$$

then $\text{sing}(S_{\mathcal{A}})$ is the image under f of all the vertices in the pre-image of $\text{sing}(\mathcal{A})$. Furthermore, recall that $f(\mathcal{V}_{\text{sing}(S_{\mathcal{A}})})$ is a set of radial arcs in $S_{\mathcal{A}}$ which are parts of \mathcal{R} .

To finish proving the statement in (3), note any quadrilateral in $\mathcal{R}^{(2)}$ whose vertices are disjoint from $\text{sing}(S_{\mathcal{A}})$ is mapped homomorphically onto a shell in $S_{\mathcal{A}}$. Since by construction the image of $\pi^{-1}(\text{sing}(\mathcal{A}))$ is precisely $\text{sing}(S_{\mathcal{A}})$, it follows that f will map the interior of each one of the rest of the quadrilaterals homomorphically onto the appropriate shell, with punctures at the corresponding vertices. This ends the proof of the proposition. \square

A geometric model to $(\mathcal{A}_{\text{sing}}, \partial\mathcal{A}_{\text{sing}}, \mathcal{T})$ is now easy to provide since the first part of (3) in the proposition above allows us to label the vertices in $\text{sing}(S_{\mathcal{A}})$ isomorphically to the labeling of the vertices in $\text{sing}(\mathcal{A})$. We will keep denoting by π the quotient map which is thereafter induced on $S_{\mathcal{A}}$. Such a quotient annulus will be called a *generalized Euclidean annulus* and will be denoted by $C_{\mathcal{A}}$. The proof of the following corollary is straightforward.

Corollary 3.27. *With the assumptions of Proposition 3.22, and with $C_{\mathcal{A}} = S_{\mathcal{A}}/\pi$, there exist*

- (1) *a tiling T of $C_{\mathcal{A}}$ by annular shells, and*
- (2) *a homeomorphism*

$$f : (\mathcal{A}_{\text{sing}}, \partial\mathcal{A}_{\text{sing}}, \mathcal{R}) \rightarrow (C_{\mathcal{A}}, \partial C_{\mathcal{A}}, T),$$

such that $f(\text{sing}(\mathcal{A})) = \text{sing}(S_{\mathcal{A}})/\pi$, f maps the interior of each quadrilateral in $\mathcal{R}^{(2)}$ onto the interior of a single annular shell in $C_{\mathcal{A}}$, f preserves the measure of each quadrilateral, i.e.,

$$\nu(R) = \mu(f(R)), \text{ for all } R \in \mathcal{R}^{(2)},$$

and f is boundary preserving.

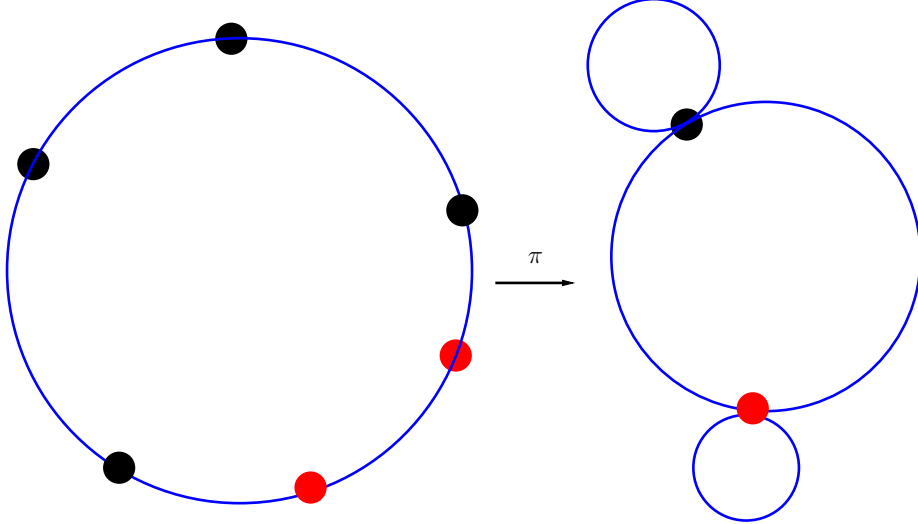


FIGURE 3.28. An example of the map π .

In the next section, we will work with a general m -connected planar domain that will be cut along singular level curves of a Dirichlet boundary value problem imposed on it. Euclidean cylinders and Euclidean cylinders with one singular boundary component will be utilized in order to allow gluing of two components of a singular level curve. To this end, recall that a conformal homeomorphism, from a concentric annulus to a Euclidean cylinder of radius equals to 1 and height equals to $\log(a/b)$, is defined by

$$(3.29) \quad F(\rho \cos(\theta), \rho \sin(\theta)) = (\cos(\theta), \sin(\theta), \log(\rho)), \quad a \leq \rho \leq b, \quad 0 < \theta \leq 2\pi,$$

where (ρ, θ) denote polar coordinates in the plane.

It easily follows that the image of an annular shell under the map F is a Euclidean rectangle. We will abuse notation and will keep the same notation for \mathcal{C}_A and its image under the mapping F .

We now define a variation of the measure ν (see equation (3.2)) in order to adjust our statements to working with such cylinders.

Definition 3.30. For any $R \in \mathcal{R}^{(2)}$, let $R_{\text{top}}, R_{\text{base}}$ be the top and base boundaries of R , respectively. Let $t \in R_{\text{top}}^{(0)}$ and $b \in R_{\text{base}}^{(0)}$ be any two vertices. Then let

$$(3.31) \quad \lambda(R) = \frac{2\pi d\theta(R_{\text{base}})}{\text{period}(\theta)} \log \frac{g(t)}{g(b)}.$$

By applying the map F and the measure λ , we may state Theorem 0.4, Proposition 3.22 and Corollary 3.27 in the language of Euclidean cylinders. We end this subsection by summarizing that in the following remark which will be applied in the next section.

Remark 3.32. Under the assumptions of Theorem 0.4, Proposition 3.22 and Corollary 3.27, all the assertions hold if one replaces S_A , generalized S_A by a Euclidean cylinder, generalized Euclidean cylinder, respectively; f by $\pi \circ F \circ f$ and an annular shell by its image under F , $F \circ \pi$, respectively, and the measure ν by the measure λ .

4. PLANAR DOMAINS OF HIGHER CONNECTIVITY

In this section we will generalize Theorem 0.4 to the case of bounded planar domains of higher connectivity. We will start by recalling an important property of the level curves of the solution of the Dirichlet boundary value problem (see Definition 1.4). This property will be essential in the proof of Theorem 4.4. In the course of the proof, we will need to know that there is a singular level curve which encloses all of the interior components of $\partial\Omega$, where Ω is the given domain. This level curve is the one along which we will cut the domain. We will keep splitting along a sequence of singular level curves in subdomains of smaller connectivity until the remaining pieces are annuli or generalized singular annuli. Once this is achieved, we will provide a gluing scheme in order to fit the pieces together in a geometric way.

Before stating the second main theorem of this paper, we need to recall a definition and a proposition. Following [2] and [23, Section 3], consider the number of sign changes in the sequence $\{f(w_1) - f(v), f(w_2) - f(v), \dots, f(w_k) - f(v), f(w_1) - f(v)\}$, which is denoted by $\text{Sgc}_f(v)$. The index of v is then defined by

$$(4.1) \quad \text{Ind}_f(v) = 1 - \frac{\text{Sgc}_f(v)}{2}.$$

Definition 4.2. A vertex whose index is different from zero will be called singular; otherwise the vertex is regular. A level set which contains at least one singular vertex will be called singular; otherwise the level set will be called regular.

The following proposition appeared as Proposition 2.28 in [18].

Proposition 4.3. *There exists a unique singular level curve which contains, in the interior of the domain it bounds, all of the inner boundary components of $\partial\Omega$.*

Such a curve will be called the *maximal* singular level curve with respect to Ω . Recall that the notion of an interior domain was discussed in Subsection 1.3.

Throughout this paper, we will not distinguish between a Euclidean rectangle and its image under an isometry. Recall (see the end of Subsection 0.3) that a singular flat, genus zero compact surface with $m > 2$ boundary components with conical singularities is called a ladder of singular pairs of pants.

We now state and prove a generalization of Theorem 0.4.

Theorem 4.4 (A Dirichlet model for an m -connected domain). *Let $(\Omega, \partial\Omega = E_1 \sqcup E_2, \mathcal{T})$ be a bounded, m -connected, planar domain with $E_2 = E_2^1 \sqcup E_2^2 \dots \sqcup E_2^{m-1}$. Let g be the solution of the Dirichlet boundary value problem defined on $(\Omega, \partial\Omega, \mathcal{T})$. Then there exist*

- (1) *a finite decomposition $\mathcal{A} = \sqcup_i \mathcal{A}_i$ of Ω , where for all i , \mathcal{A}_i is either an annulus or an annulus with one singular boundary component;*
- (2) *for all i , a cellular decomposition $\mathcal{R}_{\mathcal{A}_i}$ of \mathcal{A}_i where each 2-cell is a simple quadrilateral;*
- (3) *for all i , a finite measure λ_i defined on $\mathcal{R}_{\mathcal{A}_i}$; and*
- (4) *a ladder of singular pairs of pants S_Ω with m boundary components, such that*
 - (a) *the lengths of the m boundary components of S_Ω are determined by the Dirichlet data,*
 - (b) *there exists a decomposition of $S_\Omega = \sqcup_i C_{\mathcal{A}_i}$, where each $C_{\mathcal{A}_i}$ is either a Euclidean cylinder or a generalized Euclidean cylinder, equipped with a tiling T_i by Euclidean rectangles endowed with Lebesgue measure; and*
 - (c) *a homeomorphism*

$$f : (\Omega, \partial\Omega, \sqcup_i \mathcal{A}_i) \rightarrow (S_\Omega, \partial S_\Omega, \sqcup_i \mathcal{R}_{\mathcal{A}_i}),$$

such that f maps each \mathcal{A}_i homeomorphically onto a corresponding $C_{\mathcal{A}_i}$, and each quadrilateral in $\mathcal{R}_{\mathcal{A}_i}$ onto a rectangle in $C_{\mathcal{A}_i}$ while preserving its measure. Furthermore, f is boundary preserving (as explained in Theorem 0.4).

Proof. The first part of the proof is based on a splitting scheme along a family of singular level curves of g which will be proven to terminate after finitely many steps. We will describe in detail the first two steps of the scheme, explain why it terminates, and leave the “indices” bookkeeping required in the formal inductive step to the reader. The outcome of the first part of the proof is a scheme describing a splitting of the top domain, Ω , to simpler components, annuli and singular annuli.

The complement of $L(\Omega)$, the maximal singular curve in Ω , has m -connected components, all of which, due to Proposition 4.3, have connectivity which is smaller than $m - 1$, or are annuli, or generalized singular annuli. By the maximum principle, one of these components has all of its vertices with g -values that are greater than the g -value along $L(\Omega)$. In Subsection 1.3, such a domain was denoted by $\mathcal{O}_2(L(\Omega))$ and was called an exterior domain. Its boundary consists of E_1 and $L(\Omega)$. It follows from Proposition 4.3 and Theorem 3.19 that it is a generalized singular annulus which will be denoted by $\mathcal{A}(E_1, L(\Omega))$.

Let the full list of components be enumerated as

$$(4.5) \quad CC_1 = \{CC_{1,1}(L(\Omega)), CC_{1,2}(L(\Omega)), \dots, CC_{1,m}(L(\Omega)) = \mathcal{A}(E_1, L(\Omega))\}.$$

By definition, for each $j = 1, \dots, m - 1$, the g -value on the boundary component

$$(4.6) \quad \partial_{1,j} = \partial CC_{1,j}(L(\Omega)) \cap L(\Omega)$$

is the constant which equals the g -value on $L(\Omega)$. The other components of $\partial CC_{1,j}(L(\Omega))$, $j = 1, \dots, m-1$, are kept at g -values equal to 0. Hence, we now impose a Dirichlet boundary value problem with these values on each element in the list $CC_1 \setminus \mathcal{A}(E_1, L(\Omega))$. On $\mathcal{A}(E_1, L(\Omega))$, the induced Dirichlet boundary value problem is determined by the value of g restricted to E_1 and the value of g restricted to $L(\Omega)$. Note that imposing these boundary value problems in general will require introducing vertices of type I and of type II and changing conductance constants along new edges, as described in Subsection 1.3. These modifications are done in such a way that the restriction of the original g solves the new boundary value problems.

For $j = 1, \dots, m-1$, let k_j denote the connectivity of $CC_{1,j}(L(\Omega))$. We now repeat the procedure described in the first paragraph of the proof in each one of the connected components $CC_{1,j}(L(\Omega))$, $k_j - 2$ times, for $j = 1, \dots, m-1$, excluding those indices that correspond to annuli.

We will now describe the second step of the splitting scheme. For each $j \in \{1, \dots, m-1\}$ whose corresponding component is not an annulus, a maximal singular level curve $L_j(CC_{1,j}) = L(CC_{1,j}(L(\Omega)))$ with respect to the component $CC_{1,j}(L(\Omega))$ is chosen. This is possible because at the end of the previous step we imposed a Dirichlet boundary value problem on each one of these domains. Hence, the assertion of Proposition 4.3 and Theorem 3.19 may be applied to these domains as well.

Therefore, a new list consisting of connected components of the complement of $L_j(CC_{1,j})$ in $CC_{1,j}(L(\Omega))$, of cardinality at most $m-1$,

$$(4.7) \quad CC_{1,j} = \{CC_{1,j,1}(L_j(CC_{1,j})), CC_{1,j,2}(L_j(CC_{1,j})), \dots, CC_{1,j,m-1}(L_j(CC_{1,j}))\},$$

j as chosen above is generated. We will let the last element in this list denote the exterior domain to $L_j(CC_{1,j})$ in $CC_{1,j}$. It is, as in the first step of the scheme, a generalized singular annulus denoted by $\mathcal{A}(\partial_{1,j}, L_j(CC_{1,j}))$. The other components have connectivity which is smaller than $(m-2)$, or are annuli. Note that in this step the exterior domain from the first step in the scheme, $CC_{1,m}(L(\Omega)) = \mathcal{A}(E_1, L(\Omega))$, is left without any further splitting, since it is a generalized singular annulus.

By definition, for each j chosen as above, and each $i = 1, \dots, m-2$, the g -value on the boundary component

$$(4.9) \quad \partial_{1,j,i} = \partial CC_{1,j,i}(L_j(CC_{1,j})) \cap L_j(CC_{1,j})$$

is the constant which equals the g -value on $L_j(CC_{1,j})$. The other components of $\partial CC_{1,j,i}(L(\Omega))$ are kept at g -values equal to 0. Hence, we now impose a Dirichlet boundary value problem with the above values on each connected component in the list $CC_{1,j} \setminus \mathcal{A}(\partial_{1,j}, L_j(CC_{1,j}))$. On $\mathcal{A}(\partial_{1,j}, L_j(CC_{1,j}))$ the induced Dirichlet boundary value problem is determined by the value of g restricted to $\partial_{1,j}$ and the value of g restricted to $L_j(CC_{1,j})$. Addition of vertices of type I and II, and modifications of conductance constants will be applied in this step as in the previous one.

It follows that in each step of the splitting scheme either a domain with a fewer boundary components than the one that was split, or a singular annulus, or an annulus, is obtained. Hence, the connectivity level of each connected component after the split is either the constant number two, the constant number three, or it decreases. Therefore, the splitting scheme will terminate once all the obtained components have connectivity which equals to two or

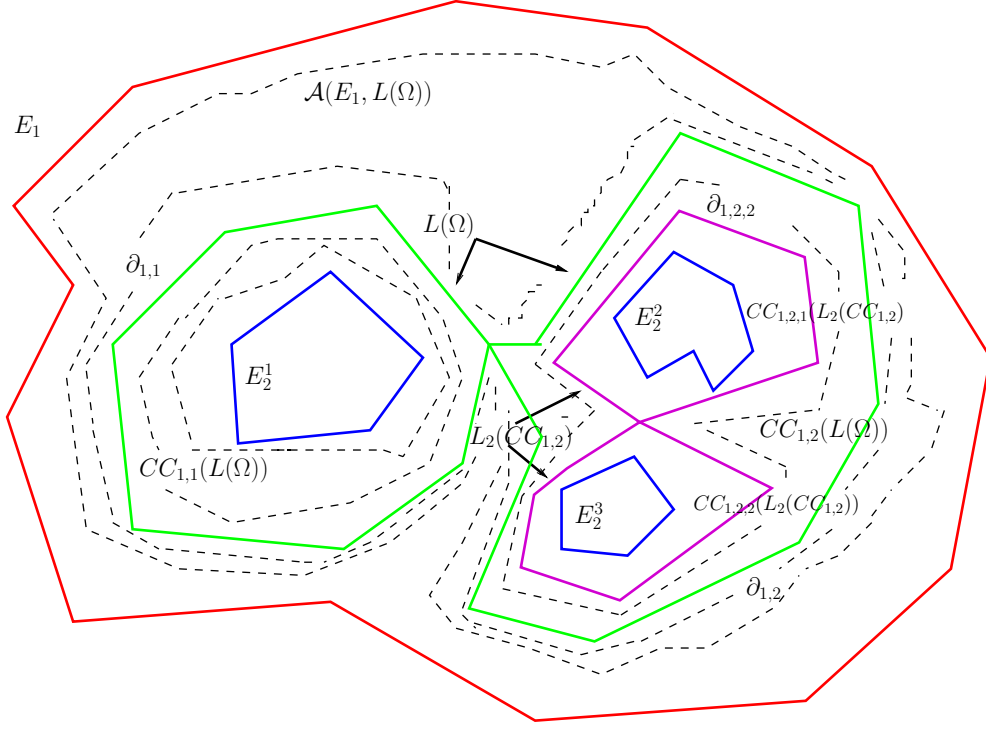


FIGURE 4.8. An example of a splitting scheme.

three, i.e., when the union of all the final generated lists is a list of lists, each containing only annuli and generalized singular annuli.

We now turn to the second part of the proof. Here, we will show that it is possible to reverse the splitting scheme, i.e., starting at the final lists generated in the splitting scheme up to the first one, CC_1 , we will glue the pieces in a *geometric way*; that is, so that the lengths of glued boundary components are equal. It is in this step where the pair-flux length will be used.

By the structure of the lists obtained in the first part of the proof, it is sufficient to show how to

- (1) glue in a geometric way elements in a list, say CC_{1,j,\dots,k_j} , that contains only annuli and a generalized singular annuli so as to form a ladder of singular pair of pants denoted by $\mathcal{S}_{1,j,\dots,k_j}$, and
- (2) glue in geometric way $\mathcal{S}_{1,j,\dots,k_j}$ to the singular boundary component of the generalized singular annulus in the list from which CC_{1,j,\dots,k_j} was formed.

Note that, if all the elements in the list CC_{1,j,\dots,k_j} are annuli, we can apply case (2), since we will map each annulus via Theorem 0.4 to a Euclidean cylinder (by first applying Theorem 0.4 and then Remark 3.32). In order to ease the notation, let us show steps (1) and (2) for the lists produced in the first part of the proof.

Fix a $j \in \{1, \dots, m-1\}$, and for $i = 1, \dots, m-2$, we apply Theorem 0.4 to $\mathcal{A}_i = CC_{1,j,i}$, which by the assumption of step (1) is an annulus. This yields a collection of concentric Euclidean annuli $\{S_i, \dots, S_{m-2}\}$ where, in the notation of Theorem 0.4, $S_i = S_{\mathcal{A}_i}$.

Furthermore, for each $i = 1, \dots, m-2$, we have

$$(4.10) \quad \{r_1^i, r_2^i\} = \{1, 2\pi \text{Length}_{(g, \theta_i)}(L(\partial_{1,j,i}))\} = \{1, 2\pi \exp\left(\frac{2\pi}{\text{period}(\theta_i)} g(\partial_{1,j,i})\right)\},$$

where θ_i is the period of the conjugate function to g , the solution of the Dirichlet boundary value problem defined on $CC_{1,j,i}$.

We now apply the map F defined in equation (3.29) to obtain a corresponding sequence of Euclidean cylinders $\{C_1, \dots, C_{m-2}\}$. All of these have radii equal to the constant 1, and their heights are given respectively by

$$(4.11) \quad h_i = \log(2\pi \text{Length}_{(g, \theta_i)}(L(\partial_{1,j,i}))) = \log 2\pi + \frac{2\pi}{\text{period}(\theta_i)} g(\partial_{1,j,i}).$$

Recall that the last component in the list $CC_{1,j}$ is the generalized singular annulus $\mathcal{A}(\partial_{1,j}, L_j(CC_{1,j}))$. Let $\theta_{1,j}$ be the conjugate harmonic function to g , the solution of the Dirichlet boundary value problem induced on it (see the paragraph preceding Proposition 3.22). Then, following Remark 3.32, we now map it to a generalized Euclidean cylinder, $C_{\mathcal{A}(\partial_{1,j}, L_j(CC_{1,j}))}$.

Measuring lengths of the image of the boundary components of $\mathcal{A}(\partial_{1,j}, L_j(CC_{1,j}))$ (using g and $\theta_{1,j}$), which are in one to one correspondence with the singular boundary of this cylinder, yields the sequence of radii

$$(4.12) \quad R_i = \frac{1}{2\pi} \frac{\text{Length}_{g, \theta_j}(\partial_{1,j,i})}{\text{Length}_{g, \theta_j}(\partial_{1,j})} = \frac{1}{2\pi} \frac{1}{\text{period}(\theta_{1,j})} \frac{\exp\left(\frac{2\pi}{\text{period}(\theta_{i,j})} g(\partial_{1,j,i})\right)}{\exp\left(\frac{2\pi}{\text{period}(\theta_{1,j})} g(\partial_{1,j})\right)} \int_{e \in \partial_{1,j,i}} d\theta_{1,j}(e),$$

for $i = 1, \dots, m-2$, where the expression on the righthand side is based on equation (2.38) and equation (2.40). The sequence R_i , $i = 1, \dots, m-2$ comprises the sequence of lengths of the round circles in the singular boundary component of $C_{\mathcal{A}(\partial_{1,j}, L_j(CC_{1,j}))}$.

Let

$$(4.13) \quad f_i = f_{R_i} = R_i z$$

be the conformal homeomorphism acting on the Euclidean cylinder $C_i = \pi \circ F \circ f(CC_{1,j,i})$, where z is the standard complex parameter on $CC_{1,j,i}$. Hence, we may glue $f_i(C_i)$ along one of its boundary components to the corresponding component in the singular boundary component of $C_{\mathcal{A}(\partial_{1,j}, L_j(CC_{1,j}))}$, so that the length of the two boundaries are the same.

This establishes step (1). We will now show how to establish step (2).

The completion of step (1) yields a ladder of singular pairs of pants which we will denote by $\mathcal{S}_{1,j}$. This ladder has $(m-1)$ components; one corresponds to $\partial_{1,j}$ and the others comprise one boundary component of each one of the cylinders $f_i \circ C_i$, $i = 1, \dots, m-2$, after these cylinders are attached. Recall that $\partial_{1,j}$ is the intersection of the generalized singular annulus $\mathcal{A}(E_1, L(\Omega))$ with the generalized singular annulus in the list $CC_{1,j}$, which is $\mathcal{A}(\partial_{1,j}, L_j(CC_{1,j}))$.

Let g be the solution of the induced Dirichlet boundary value problem on $\mathcal{A}(E_1, L(\Omega))$ and let θ be the harmonic conjugate function to g (see the paragraph preceding Proposition 3.22).

Let

$$(4.14) \quad \tau_j = \frac{l_j}{\text{Length}_{g,\theta}(\partial_{i,j})},$$

where l_j denotes the length of boundary component that corresponds to $\partial_{1,j}$ in $\mathcal{S}_{1,j}$, and $C_{\mathcal{A}(E_1, L(\Omega))}$ is the generalized Euclidean cylinder constructed for $\mathcal{A}(E_1, L(\Omega))$ (see Remark 3.32). After applying a conformal expansion of magnitude τ_j to $\mathcal{S}_{1,j}$, it then may be glued along this boundary component to $C_{\mathcal{A}(E_1, L(\Omega))}$ in such a way that the length of the corresponding circle in the round bouquet $\partial C_{\mathcal{A}(E_1, L(\Omega))}$ has the same length. This establishes step (2).

By construction, it is clear that the pair-flux length of the boundary component of S_Ω that corresponds to E_1 is equal to

$$(4.15) \quad 2\pi \exp\left(\frac{2\pi}{\text{period}(\theta)}k\right),$$

where θ is the harmonic conjugate to g , the solution of the Dirichlet boundary value problem induced on $\mathcal{A}(E_1, L(\Omega))$ (recall from Definition 1.4 that k is the value of g restricted to E_1). The lengths of the remaining $(m-1)$ boundary components of S_Ω which correspond to the $m-1$ boundary components of E_2 are determined by the process described in the previous part of the proof.

The length of a component in S_Ω which corresponds to E_2^i , $i \in \{1, \dots, m-1\}$, measured with respect to the pair-flux metric, is obtained by successively multiplying a sequence of ratios of lengths. These ratios are uniquely determined as in equation (4.12) and equation (4.14), and present the expansion factor needed in order to match the gluing of a (generalized) cylinder to the one which induced it in the splitting process.

Cone angles are formed whenever more than two cylinders meet at a vertex; viewed in Ω , this will occur whenever more than two circles in a generalized bouquet meet at a vertex. The computation of the cone angles is solely determined by g and \mathcal{T} . This analysis first appeared in Theorem 0.4 in [18]. Specifically, the cone angle $\phi(v)$ at a singular vertex v , which is the unique tangency point of $n+1$ Euclidean cylinders, satisfies

$$(4.16) \quad \phi(v) = 2(n+1)\pi.$$

The proof of the theorem is thus complete, with f defined to be the union of the individual maps constructed at each stage.

□

Remark 4.17. There is a technical difficulty in our construction if some pair of adjacent vertices of $\mathcal{T}^{(0)}$ has the same g -values (the first occurrence is in equation (4.1)). This issue has already been discussed and resolved in [18, Remark 4.14].

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